Appendix:

This appendix provides the mathematical details of the STEP-3: Modeling Decision-Making described above. When we say "will", we mean our own choice. In other words, if the reality is to be done stochastically, the object must have said "will" to the expected value from the probability space. First, we focus on will as an expectation, and for modeling purposes, we define the "hypothetical dynamics" of will. We think of volition as the process of changing the initial state probability space to create a new state probability space. First, we define position x as the Kullback–Leibler divergence D_{KL} between the probability space representing the initial state and the future probability space realized by the will. This distance is not limited to the Kullback–Leibler divergence, but may be any other distance. We believe that D_{KL} is a measure of the difference between probability distributions, and if we follow its time change, it can be perceived as a change in position in normal dynamics. In addition, the time derivative of x corresponds to velocity and can be interpreted as the speed of change in the probability distribution, and the second derivative of time can also be considered as acceleration α. Furthermore, consider, perhaps intuitively, the "inertia" of the will. There is a lag in remembering when we must remember something and act on it. We will call this degree of slow speed "inertia" and that degree the mass m of decision making. The speed and acceleration defined above are the processing speed to the decision after the recall, and are considered as independent parameters of m. In this paper, we consider decision making as a selection from past behavioral information from past episodic memory, and the acceleration α, mass m defined above should be closely connected to past episodic memory. Then, again in analogy to dynamics, we define the potential V of past episodic memory as a function to satisfy:

$$-\frac{\partial V}{\partial x} = m\alpha \qquad \text{(A1)}$$

Now let us assume that $m \times \alpha$ represents a force in ordinary dynamics, but within Eq. (A1), we understand it as a relation defining the potential V, and there is no further meaning. It is natural to think that past episodic memory influences the speed and acceleration of the will and introduces potential as an indicator. All the above definitions of hypothetical dynamics are for will as expectation. In contrast, we think that in actual decision making, randomness is added by the state of the brain - the state of neurons, etc. That is, in Brownian motion, the same image as fine particles being shaken by surrounding liquid molecules. In fact, the velocity of Brownian moving particles varies randomly, so that dx/dt cannot be defined for each instant. Similarly, in decisions involving randomness that is not an expectation, it is impossible to define such things as speed as the expectation mentioned above. Therefore, we define the time variation of position x as follows.

$$x(t+\Delta t) - x(t) = b(x(t),t)\Delta t + w(t+\Delta t) - w(t) \qquad \text{(A2)}$$

where w(t) is the Wiener process. This represents that the time variation of the Kullback–Leibler divergence changes under the influence of the Wiener process. Based on (A2), consider calculating the velocity and acceleration in decision making as expected values. So, when there is a random variable f(t) that generally depends on time t, its 〈forward mean derivative〉 is defined below:

$$D[f(t)] \equiv \lim_{\Delta t \to 0} \left\langle \frac{f(t+\Delta t)-f(t)}{\Delta t} \middle| fixed\ f(s) \quad (s \le t) \right\rangle \qquad \text{(A3)}$$

The 〈 | 〉 on the right-hand side represents the conditional time average that f(s) before t is fixed. In this (A2), the forward average differential coefficient of x(t) is obtained as follows:

$$D[f(t)] \equiv b(x(t),t) \qquad \text{(A4)}$$

Next, 〈Backward mean derivative〉 is defined below.

$$D_*[f(t)] \equiv \lim_{\Delta t \to 0} \left\langle \frac{f(t)-f(t-\Delta t)}{\Delta t} \middle| fixed\ x(s) \quad (s \le t) \right\rangle \quad =0 \qquad \text{(A5)}$$

Using Eq. (A5) to determine the backward-facing mean derivative of x (t), we obtain:

$$D_*[f(t)] \equiv b_*(x(t), t) \tag{A6}$$

Using this b_*, we obtain:

$$x(t) - x(t - \Delta t) \equiv b_*(x(t), t) + w(t) - w(t - \Delta t) \tag{A7}$$

The acceleration α at the expected value is then defined below.

$$\alpha(t) = \frac{1}{2}(D_*D + DD_*)x(t) \tag{A8}$$

Calculating the right-hand second term DD_* of the above equation, we obtain:

$$DD_*x(t) = \frac{\partial b_*}{\partial t} + b\frac{\partial b_*}{\partial x} + \frac{\nu}{2}\frac{\partial^2 b_*}{\partial x^2} \tag{A9}$$

Similarly, D_*D is:

$$D_*Dx(t) = \frac{\partial b}{\partial t} + b_*\frac{\partial b_*}{\partial x} + \frac{\nu}{2}\frac{\partial^2 b}{\partial x^2} \tag{A10}$$

Next, the following variables are introduced:

$$u = \frac{1}{2}(b - b^*), v = \frac{1}{2}(b + b^*) \tag{A11}$$

Using this, the expected acceleration α is:

$$\alpha = -\frac{\nu}{2}\frac{\partial^2 u}{\partial x^2} + v\frac{\partial v}{\partial x} - u\frac{\partial u}{\partial x} + \frac{\partial v}{\partial t} \tag{A12}$$

Applying (A1) to this α, we obtain the following relation:

$$\frac{\partial v}{\partial t} = \frac{\nu}{2}\frac{\partial^2 u}{\partial x^2} - v\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} - \frac{1}{m}\frac{\partial V}{\partial x} \tag{A13}$$

If the above definitions are allowed, the theory that E. Nelson tried to explain the governing equations of quantum mechanics based on the Wiener process is applicable as is. The following summarizes the formulation by E. Nelson. First, the following Fokker-Planck equation is used.

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x}b(x, t) - \frac{\nu}{2}\frac{\partial^2}{\partial x^2}\right]\rho(x_0, t_0|x, t) = 0 \tag{A14}$$

Where, t is time, x is defined as Kullback–Leibler divergence. The second term on the left hand side has the following meanings:

$$\frac{\partial}{\partial x}[b(x, t)\rho] \tag{A15}$$

Here, since no transition can occur without time, at $t = t_0$ the following is satisfied, which is the initial condition:

$$\rho(x_0, t_0|x, t) = \delta(x - x_0) \tag{A16}$$

Moreover, ρ satisfies the following conditions.

$$\int \rho(x_0, t_0|x, t)\, dx = 1 \tag{A17}$$

In Eq. (A14), the equation with time reversed is as follows:

$$\left[-\frac{\partial}{\partial t} + \frac{\partial}{\partial x}b^*(x, t) + \frac{\nu}{2}\frac{\partial^2}{\partial x^2}\right]\rho = 0 \tag{A18}$$

By summing (A14) and (A16), we obtain:

$$\frac{\partial}{\partial x}\left[-(b - b^*) + \nu\frac{\partial^2}{\partial x^2}\right]\rho = 0 \tag{A19}$$

Using the findings from (A19) and (A11), we obtain the following:

$$u = \frac{\nu}{2}\frac{1}{\rho}\frac{\partial \rho}{\partial x} = \frac{\nu}{2}\frac{\partial}{\partial x} ln(\rho) \tag{A20}$$

Replacing (A14) and (A18) with the expression of v yields:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(v\rho) = 0 \tag{A21}$$

Erasing ρ using (A21) and (A22) yields the following equation for only u, v:

$$\frac{\partial u}{\partial t} = -\frac{\nu}{2}\frac{\partial^2 v}{\partial x^2} - \frac{\partial}{\partial x}(uv) \tag{A22}$$

Eq. (A13) and (A22) are written together as follows:

$$\frac{\partial u}{\partial t} = -\frac{\nu}{2}\frac{\partial^2 v}{\partial x^2} - v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x} \tag{A23}$$

$$\frac{\partial v}{\partial t} = \frac{\nu}{2}\frac{\partial^2 u}{\partial x^2} - v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x} - \frac{1}{m}\frac{\partial V}{\partial x} \tag{A24}$$

We define the function χ below.

$$\chi(x,t) = u(x,t) + iv(x,t) \tag{A25}$$

where i is an imaginary unit. Using this χ, (A24) and (A25) are unified as follows:

$$-i\frac{\partial \chi}{\partial t} = \frac{\nu}{2}\frac{\partial^2 \chi}{\partial x^2} + \frac{1}{2}\frac{\partial^2 \chi}{\partial x^2} - \frac{1}{m}\frac{\partial V}{\partial x} \tag{A26}$$

In addition, the following transformations are performed.

$$\chi = \frac{\nu}{2}\frac{1}{\Psi}\frac{\partial \psi}{\partial x} = \frac{\nu}{2}\frac{\partial}{\partial x} ln(\Psi) \tag{A27}$$

The characteristics of this transformation are described below. The characteristics of this transformation are described below.

a) time derivative of equation (A27):

$$-i\frac{\partial \chi}{\partial x} = -i\nu\frac{\partial}{\partial x}\frac{\partial}{\partial t} ln(\Psi) = -i\nu\frac{\partial}{\partial x}\left(\frac{1}{\Psi}\frac{\partial \Psi}{\partial t}\right) \tag{A28}$$

b) spatial derivative of equation (A28):

$$\nu\frac{\partial \chi}{\partial x} = -\frac{\nu^2}{\psi^2}\left(\frac{\partial \psi}{\partial x}\right)^2 + \frac{\nu^2}{\Psi}\frac{\partial^2 \psi}{\partial x^2} \tag{A29}$$

Taking advantage of the fact that the first term on the right-hand side is $-\chi^2$, further differentiation yields:

$$\frac{\nu}{2}\frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2}\frac{\partial \chi^2}{\partial x} = \frac{\nu^2}{2}\frac{\partial}{\partial x}\left[\frac{1}{\Psi}\frac{\partial^2 \psi}{\partial x^2}\right] \tag{A30}$$

Substituting (A26), (A27) for (A28) yields:

$$\frac{\partial}{\partial x}\left[i\nu\frac{1}{\Psi}\frac{\partial \Psi}{\partial t} + \frac{\nu^2}{2}\frac{1}{\Psi}\frac{\partial^2 \Psi}{\partial x^2} - \frac{1}{m}V\right] = 0 \tag{A31}$$

Since this expression means that the values in [ ] are independent of the space x, we put it equal to the time-only function, η (t). Then we get:

$$i\nu\frac{\partial \Psi}{\partial t} = \left[-\frac{\nu^2}{2}\frac{\partial^2 \Psi}{\partial x^2} + \frac{1}{m}V + \eta\right]\Psi \tag{A32}$$

Furthermore, we transform Ψ using ψ and η as follows:

$$\Psi(x,t) = \psi(x,t) exp\left(-\frac{i}{\nu}\int \eta(s)ds\right) \tag{A33}$$

In this transformation, η (t) is removed and finally becomes an equation of only Ψ below.

$$i\nu\frac{\partial \psi}{\partial t} = \left[-\frac{\nu^2}{2}\frac{\partial^2 \psi}{\partial x^2} + \frac{1}{m}V\right]\psi(x,t) \tag{A34}$$